Travelling wave solutions of BBM-like equations by means of factorization

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Abstract

In this work, we apply the factorization technique to the Benjamin-Bona-Mahony like equations in order to get travelling wave solutions. We will focus on some special cases for which $m \neq n$, and we will obtain these solutions in terms of Weierstrass functions.

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1 Introduction

In this paper, we will consider the Benjamin-Bona-Mahony (BBM) [1] like equation ($B(m,n)$) with a fully nonlinear dispersive term of the form

$$u_t + u_x + a\,(u^m)_x - (u^n)_{xxt} = 0, \qquad m,\, n > 1,\ m \neq n\,. \tag{1}$$

This equation is similar to the nonlinear dispersive equation $K(m,n)$,

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \qquad m > 0,\, 1 < n \leq 3 \tag{2}$$

which has been studied in detail by P. Rosenau and J.M. Hyman [2]. In the literature there are many studies dealing with the travelling wave solutions of the $K(m,n)$ and $B(m,n)$ equations, but in general they are restricted to the case $m = n$ [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. When $m \neq n$, the solutions of $K(m,n)$ were investigated in [2, 3]. Our aim here is just to search for solutions of the equations $B(m,n)$, with $m \neq n$, by means of the factorization method.

We remark that this method [15, 16, 17, 18], when it is applicable, allows to get directly and systematically a wide set of solutions, compared with other methods used in the BBM equations. For example, the direct integral method used by C. Liu [19] can only be applied to the $B(2,1)$ equation. However, the factorization technique can be used to more equations than the direct integral method and also, in some cases, it gives rise to more general solutions than the sine-cosine and the tanh methods [11, 12, 20]. This factorization approach to find travelling wave solutions of nonlinear equations has been extended to third order nonlinear ordinary differential equations (ODE's) by D-S. Wang and H. Li [21].

When we look for the travelling wave solutions of Eq. (1), first we reduce the form of the $B(m,n)$ equation to a second order nonlinear ODE and then, we can immediately apply factorization technique. Here, we will assume $m \neq n$, since the case $m = n$ has already been examined in a previous article following this method [14].

This paper is organized as follows. In section 2 we introduce factorization technique for a special type of the second order nonlinear ODE's. Then, we apply straightforwardly the factorization to the related second order nonlinear ODE to get travelling wave solutions of $B(m,n)$ equation in section 3. We obtain the solutions for these nonlinear ODE's and the $B(m,n)$ equation in terms of Weierstrass functions in section 4. Finally, in section 5 we will add some remarks.

2 Factorization of nonlinear second order ODE's

Let us consider, the following nonlinear second order ODE

$$\frac{d^2W}{d\theta^2} - \beta\frac{dW}{d\theta} + F(W) = 0 \tag{3}$$

where β is constant and $F(W)$ is an arbitrary function of W. The factorized form of this equation can be written as

$$\left[\frac{d}{d\theta} - f_2(W,\theta)\right]\left[\frac{d}{d\theta} - f_1(W,\theta)\right]W(\theta) = 0\,. \tag{4}$$

Here, f_1 and f_2 are unknown functions that may depend explicitly on W and θ. Expanding (4) and comparing with (3), we obtain the following consistency conditions

$$f_1 f_2 = \frac{F(W)}{W} + \frac{\partial f_1}{\partial\theta}, \qquad f_2 + \frac{\partial(W f_1)}{\partial W} = \beta. \tag{5}$$

If we solve (5) for f_1 or f_2, it will supply us to write a compatible first order ODE

$$\left[\frac{d}{d\theta} - f_1(W,\theta)\right]W(\theta) = 0 \tag{6}$$

that provides a solution for the nonlinear ODE (3) [15, 16, 17, 18]. In the applications of this paper f_1 and f_2 will depend only on W.

3 Factorization of the BBM-like equations

When Eq. (1) has the travelling wave solutions in the form

$$u(x,t) = \phi(\xi), \qquad \xi = hx + wt \tag{7}$$

where h and w are real constants, substituting (7) into (1) and after integrating, we get the reduced form of Eq. (1) to the second order nonlinear ODE

$$(\phi^n)_{\xi\xi} - A\,\phi - B\,\phi^m + D = 0\,. \tag{8}$$

Notice that the constants in Eq. (8) are

$$A = \frac{h+w}{h^2\,w}, \qquad B = \frac{a}{h\,w}, \qquad D = \frac{R}{h^2\,w} \tag{9}$$

and R is an integration constant. Now, if we introduce the following natural transformation of the dependent variable

$$\phi^n(\xi) = W(\theta), \qquad \xi = \theta \tag{10}$$

Eq. (8) becomes

$$\frac{d^2W}{d\theta^2} - A\,W^{\frac{1}{n}} - B\,W^{\frac{m}{n}} + D = 0. \tag{11}$$

Now, we can apply the factorization technique to Eq. (11). Comparing Eq. (3) and Eq. (11), we have $\beta = 0$ and

$$F(W) = -(A\,W^{\frac{1}{n}} + B\,W^{\frac{m}{n}} - D)\,. \tag{12}$$

Then, from (5) we get only one consistency condition

$$f_1^2 + f_1\,W\frac{df_1}{dW} - A\,W^{\frac{1-n}{n}} - B\,W^{\frac{m-n}{n}} + D\,W^{-1} = 0 \tag{13}$$

whose solutions are

$$f_1(W) = \pm\frac{1}{W}\sqrt{\frac{2\,n\,A}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,n\,B}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W + C} \tag{14}$$

where C is an integration constant. Thus, the first order ODE (6) takes the form

$$\frac{dW}{d\theta} \mp \sqrt{\frac{2\,n\,A}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,n\,B}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W + C} = 0\,. \tag{15}$$

In order to solve this equation for W in a more general way, let us take W in the form $W = \varphi^p$, $p \neq 0, 1$, then, the first order ODE (15) is rewritten in terms of φ as

$$(\frac{d\varphi}{d\theta})^2 = \frac{2\,n\,A}{p^2\,(n+1)}\,\varphi^{p(\frac{1-n}{n})+2} + \frac{2\,n\,B}{p^2\,(m+n)}\,\varphi^{p(\frac{m-n}{n})+2} - \frac{2\,D}{p^2}\,\varphi^{2-p} + \frac{C}{p^2}\,\varphi^{2-2\,p}\,. \tag{16}$$

If we want to guarantee the integrability of (16), the powers of φ have to be integer numbers between 0 and 4 [24]. Having in mind the conditions on n, m ($n \neq m > 1$) and p ($p \neq 0$), we have the following possible cases:

- If $C = 0,\ D = 0$, we can choose p and m in the following way

$$p = \pm\frac{2n}{1-n} \quad \text{with} \quad m = \frac{n+1}{2}, \frac{3\,n-1}{2}, 2\,n-1 \tag{17}$$

  and

$$p = \pm\frac{n}{1-n} \quad \text{with} \quad m = 2\,n-1, 3\,n-2\,. \tag{18}$$

  It can be checked that the two choices of sign in (17) and (18) give rise to the same solutions for Eq. (1). Therefore, we will consider only one of them. Then, taking $p = -\frac{2n}{1-n}$, Eq. (16) becomes

$$(\frac{d\varphi}{d\theta})^2 = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{n\,(3\,n+1)}\,\varphi, \quad m = \frac{n+1}{2} \tag{19}$$

$$(\frac{d\varphi}{d\theta})^2 = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{n\,(5\,n-1)}\,\varphi^3, \quad m = \frac{3\,n-1}{2} \tag{20}$$

$$(\frac{d\varphi}{d\theta})^2 = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{n\,(3\,n-1)}\,\varphi^4, \quad m = 2\,n-1 \tag{21}$$

  and for $p = -\frac{n}{1-n}$,

$$(\frac{d\varphi}{d\theta})^2 = \frac{2\,A\,(n-1)^2}{n\,(n+1)}\,\varphi + \frac{2\,B\,(n-1)^2}{n\,(3\,n-1)}\,\varphi^3, \quad m = 2\,n-1 \tag{22}$$

$$(\frac{d\varphi}{d\theta})^2 = \frac{2\,A\,(n-1)^2}{n\,(n+1)}\,\varphi + \frac{B\,(n-1)^2}{n\,(2\,n-1)}\,\varphi^4, \quad m = 3\,n-2\,. \tag{23}$$

- If $C = 0$, we have the special cases, $p = \pm 2$, $n = 2$ with $m = 3, 4$.

  Due to the same reason in the above case, we will consider only $p = 2$. Then, Eq. (16) takes the form:

$$(\frac{d\varphi}{d\theta})^2 = -\frac{D}{2} + \frac{A}{3}\,\varphi + \frac{B}{5}\,\varphi^3, \quad m = 3 \tag{24}$$

$$(\frac{d\varphi}{d\theta})^2 = -\frac{D}{2} + \frac{A}{3}\,\varphi + \frac{B}{6}\,\varphi^4, \quad m = 4\,. \tag{25}$$

- If $A = C = 0$, we have $p = \pm 2$ with $m = \dfrac{n}{2}, \dfrac{3\,n}{2}, 2\,n$.

  In this case, for $p = 2$, Eq. (16) has the following form:

$$(\frac{d\varphi}{d\theta})^2 = -\frac{D}{2}\,\varphi^4 + \frac{B}{3}\,\varphi^3, \quad m = \frac{n}{2} \tag{26}$$

$$(\frac{d\varphi}{d\theta})^2 = -\frac{D}{2}\,\varphi^4 + \frac{B}{5}\,\varphi, \quad m = \frac{3\,n}{2} \tag{27}$$

$$(\frac{d\varphi}{d\theta})^2 = -\frac{D}{2}\,\varphi^4 + \frac{B}{6}, \quad m = 2\,n\,. \tag{28}$$

- If $A = 0$, we have $p = \pm 1$ with $m = 2\,n, 3\,n$.

  Here, also we will take only the case $p = 1$, then, we will have the equations:

$$(\frac{d\varphi}{d\theta})^2 = -2\,D\,\varphi + \frac{2}{3}\,B\,\varphi^3 + C\varphi^4, \quad m = 2n \tag{29}$$

$$(\frac{d\varphi}{d\theta})^2 = -2\,D\,\varphi + \frac{B}{2}\,\varphi^4 + C, \quad m = 3n\,. \tag{30}$$

- If $A = D = 0$, we have $p = \pm\dfrac{1}{2}$ with $m = 3\,n, 5\,n$.

  Thus, for $p = \dfrac{1}{2}$, Eq. (16) becomes:

$$(\frac{d\varphi}{d\theta})^2 = 2\,B\,\varphi^3 + 4\,C\varphi, \quad m = 3n \tag{31}$$

$$(\frac{d\varphi}{d\theta})^2 = \frac{4}{3}\,B\,\varphi^4 + 4\,C\,\varphi, \quad m = 5n\,. \tag{32}$$

4 Travelling wave solutions for BBM-like equations

In this section, we will obtain the solutions of the differential equations (19)-(23) in terms of Weierstrass function, $\wp(\theta; g_2, g_3)$, which allow us to get the travelling wave solutions of $B(m,n)$ equations (1). The rest of equations (24)-(32) can be dealt with a similar way, but they will not be worked out here for the sake of shortness.

First, we will give some properties of the $\wp$ function which will be useful in the following [25, 26].

4.1 Relevant properties of the $\wp$ function

Let us consider a differential equation with a quartic polynomial

$$\Big(\frac{d\varphi}{d\theta}\Big)^2 = P(\varphi) = a_0\,\varphi^4 + 4\,a_1\,\varphi^3 + 6\,a_2\,\varphi^2 + 4\,a_3\,\varphi + a_4\,. \tag{33}$$

The solution of this equation can be written in terms of the Weierstrass function where the invariants g_2 and g_3 of (33) are

$$g_2 = a_0\,a_4 - 4\,a_1\,a_3 + 3\,a_2^2, \;\; g_3 = a_0\,a_2\,a_4 + 2\,a_1\,a_2\,a_3 - a_2^3 - a_0\,a_3^2 - a_1^2\,a_4 \tag{34}$$

and the discriminant is given by $\Delta = g_2^3 - 27\,g_3^2$. Then, the solution φ can be found as

$$\varphi(\theta) = \varphi_0 + \frac{1}{4}P_\varphi(\varphi_0)\left(\wp(\theta; g_2, g_3) - \frac{1}{24}P_{\varphi\varphi}(\varphi_0)\right)^{-1} \tag{35}$$

where the subindex in $P_\varphi(\varphi_0)$ denotes the derivative with respect to φ, and φ_0 is one of the roots of the polynomial $P(\varphi)$ (33). Depending of the selected root φ_0, we will have a solution with a different behavior [14].

Here, also we want to recall some other properties of the Weierstrass functions [27]:

i) The case $g_2 = 1$ and $g_3 = 0$ is called lemniscatic case

$$\wp(\theta; g_2, 0) = g_2^{1/2}\, \wp(\theta\, g_2^{1/4}; 1, 0), \qquad g_2 > 0 \tag{36}$$

ii) The case $g_2 = -1$ and $g_3 = 0$ is called pseudo-lemniscatic case

$$\wp(\theta; g_2, 0) = |g_2|^{1/2}\, \wp(\theta\, |g_2|^{1/4}; -1, 0), \qquad g_2 < 0 \tag{37}$$

iii) The case $g_2 = 0$ and $g_3 = 1$ is called equianharmonic case

$$\wp(\theta; g_2, 0) = g_3^{1/3}\, \wp(\theta\, g_3^{1/6}; 0, 1), \qquad g_3 > 0\,. \tag{38}$$

Once obtained the solution $W(\theta)$, taking into account (7), (10) and $W = \varphi^p$, the solution of Eq. (1) is obtanied as

$$u(x,t) = \phi(\xi) = W^{\frac{1}{n}}(\theta) = \varphi^{\frac{p}{n}}(\theta), \qquad \theta = \xi = h\,x + w\,t. \tag{39}$$

4.2 The case $C = 0,\ D = 0$, $p = -\dfrac{2n}{1-n}$

- $m = \dfrac{n+1}{2}$

  Equation (19) can be expressed as

$$(\frac{d\varphi}{d\theta})^2 = P(\varphi) = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{n\,(3\,n+1)}\,\varphi \tag{40}$$

  and from $P(\varphi) = 0$, we get the root of this polynomial

$$\varphi_0 = -\frac{A\,(3\,n+1)}{2\,B\,(n+1)}\,. \tag{41}$$

  The invariants (34) are: $g_2 = g_3 = 0$, and $\Delta = 0$. Therefore, having in mind $\wp(\theta; 0, 0) = \dfrac{1}{\theta^2}$, we can find the solution of (19) from (35) for φ_0, given by (41),

$$\varphi(\theta) = \frac{B^2\,(n-1)^2\,(n+1)\,\theta^2 - 2\,A\,n\,(3\,n+1)^2}{4\,B\,n\,(n+1)\,(3\,n+1)}\,. \tag{42}$$

  Now, the solution of Eq. (1) reads from (39)

$$u(x,t) = \left[\frac{B^2\,(n-1)^2\,(n+1)\,(h\,x + w\,t)^2 - 2\,A\,n\,(3\,n+1)^2}{4\,B\,n\,(n+1)\,(3\,n+1)}\right]^{\frac{2}{n-1}}\,. \tag{43}$$

- $m = \dfrac{3\,n-1}{2}$

  In this case, our equation to solve is (20) and the polynomial has the form

$$P(\varphi) = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{n\,(5\,n-1)}\,\varphi^3 \tag{44}$$

with one real root: $\varphi_0 = \left(\frac{-A\,(5\,n-1)}{2\,B\,(n+1)}\right)^{1/3}$. Here, the discriminant is different from zero with the invariants

$$g_2 = 0, \qquad g_3 = \frac{-A\,B^2\,(n-1)^6}{32\,n^3\,(n+1)\,(5\,n-1)^2}\,. \tag{45}$$

Then, the solution of (20) is obtained by (35) for φ_0,

$$\varphi = \varphi_0\,\left[\frac{4\,n\,(5\,n-1)\,\wp(\theta;0,g_3) + 2\,B\,(n-1)^2\,\varphi_0}{4\,n\,(5\,n-1)\,\wp(\theta;0,g_3) - B\,(n-1)^2\,\varphi_0}\right] \tag{46}$$

and we get the solution of Eq. (1) from (39) as

$$u(x,t) = \left[\varphi_0^2\,\left(\frac{4\,n\,(5\,n-1)\,\wp(h\,x+w\,t;0,g_3) + 2\,B\,(n-1)^2\,\varphi_0}{4\,n\,(5\,n-1)\,\wp(h\,x+w\,t;0,g_3) - B\,(n-1)^2\,\varphi_0}\right)^2\right]^{\frac{1}{n-1}} \tag{47}$$

with the conditions: $A<0, g_3>0$, for $\varphi_0 = \left(\frac{-A\,(5\,n-1)}{2\,B\,(n+1)}\right)^{1/3}$. Using the relation (38), we can write the solution (47) in terms of equianharmonic case of the Weierstrass function:

$$u(x,t) = \left[\left(\frac{-A\,(5\,n-1)}{2\,B\,(n+1)}\right)^{2/3}\,\left(\frac{2^{2/3}\,\wp((h\,x+w\,t)\,g_3^{1/6};0,1)+2}{2^{2/3}\,\wp((h\,x+w\,t)\,g_3^{1/6};0,1)-1}\right)^2\right]^{\frac{1}{n-1}}\,. \tag{48}$$

- $m = 2\,n-1$

In Eq. (21), the quartic polynomial is

$$P(\varphi) = \frac{A\,(n-1)^2}{2\,n\,(n+1)} + \frac{B\,(n-1)^2}{2\,n\,(3\,n-1)}\,\varphi^4 \tag{49}$$

and has two real roots: $\varphi_0 = \pm\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/4}$ for $A<0,\ B>0$ or $A>0,\ B<0$. In this case, the invariants are

$$g_2 = \frac{A\,B\,(n-1)^4}{4\,n^2\,(n+1)\,(3\,n-1)}, \qquad g_3 = 0\,. \tag{50}$$

Here, also the discriminant is different from zero, $\Delta \neq 0$. We obtain the solution of (21) from (35) for φ_0,

$$\varphi = \varphi_0\,\left[\frac{4\,n\,(n+1)\,\varphi_0^2\,\wp(\theta;g_2,0) - A\,(n-1)^2}{4\,n\,(n+1)\,\varphi_0^2\,\wp(\theta;g_2,0) + A\,(n-1)^2}\right] \tag{51}$$

and we get the solution of Eq. (1) from (39) as

$$u(x,t) = \left[\varphi_0^2\,\left(\frac{4\,n\,(n+1)\,\varphi_0^2\,\wp(h\,x+w\,t;g_2,0) - A\,(n-1)^2}{4\,n\,(n+1)\,\varphi_0^2\,\wp(h\,x+w\,t;g_2,0) + A\,(n-1)^2}\right)^2\right]^{\frac{1}{n-1}} \tag{52}$$

with the conditions for real solutions: $A<0,\ B>0,\ g_2<0$ or $A>0,\ B<0,\ g_2<0$.

Having in mind the relation (37), the solution (52) can be expressed in terms of the pseudo-lemniscatic case of the Weierstrass function:

$$u(x,t) = \left[\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}\,\left(\frac{2\,\wp((h\,x+w\,t)|g_2|^{1/4};-1,0)+1}{2\,\wp((h\,x+w\,t)|g_2|^{1/4};-1,0)-1}\right)^2\right]^{\frac{1}{n-1}} \tag{53}$$

for $A<0,\ B>0,\ g_2<0$ and

$$u(x,t) = \left[\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}\,\left(\frac{2\,\wp((h\,x+w\,t)|g_2|^{1/4};-1,0)-1}{2\,\wp((h\,x+w\,t)|g_2|^{1/4};-1,0)+1}\right)^2\right]^{\frac{1}{n-1}} \tag{54}$$

for $A>0,\ B<0,\ g_2<0$.

4.3 The case $C = 0,\ D = 0$, $p = -\dfrac{n}{1-n}$

- $m = 2\,n - 1$

  Now, the polynomial is cubic

$$P(\varphi) = \frac{2\,A\,(n-1)^2}{n\,(n+1)}\,\varphi + \frac{2\,B\,(n-1)^2}{n\,(3\,n-1)}\,\varphi^3 \tag{55}$$

and has three distinct real roots: $\varphi_0 = 0$ and $\varphi_0 = \pm\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}$ for $A < 0,\ B > 0$ or $A > 0,\ B < 0$. Now, the invariants are

$$g_2 = \frac{-A\,B\,(n-1)^4}{n^2\,(n+1)\,(3\,n-1)}, \qquad g_3 = 0 \tag{56}$$

and $\Delta \neq 0$. The solution of (22) is obtained from (35) for φ_0,

$$\varphi = \varphi_0\,\left[\frac{2\,n\,(n+1)\,\varphi_0\,\wp(\theta; g_2, 0) - A\,(n-1)^2}{2\,n\,(n+1)\,\varphi_0\,\wp(\theta; g_2, 0) + A\,(n-1)^2}\right] \tag{57}$$

and substituting (57) in (39), we get the solution of Eq. (1) as

$$u(x,t) = \left[\varphi_0\,\left(\frac{2\,n\,(n+1)\,\varphi_0\,\wp(h\,x + w\,t; g_2, 0) - A\,(n-1)^2}{2\,n\,(n+1)\,\varphi_0\,\wp(h\,x + w\,t; g_2, 0) + A\,(n-1)^2}\right)\right]^{\frac{1}{n-1}} \tag{58}$$

with the conditions: $A < 0,\ B > 0,\ g_2 > 0$ and $A > 0,\ B < 0,\ g_2 > 0$ for $\varphi_0 = \left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}$. While the root $\varphi_0 = 0$ leads to the trivial solution, $u(x,t) = 0$, the other root $\varphi_0 = -\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}$ gives rise to imaginary solutions.

Now, we can rewrite the solution (58) in terms of the lemniscatic case of the Weierstrass function using the relation (36) in (58):

$$u(x,t) = \left[\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}\,\left(\frac{2\,\wp((h\,x + w\,t)\,g_2^{1/4}; 1, 0) + 1}{2\,\wp((h\,x + w\,t)\,g_2^{1/4}; 1, 0) - 1}\right)\right]^{\frac{1}{n-1}} \tag{59}$$

for $A < 0,\ B > 0,\ g_2 > 0$ and

$$u(x,t) = \left[\left(\frac{-A\,(3\,n-1)}{B\,(n+1)}\right)^{1/2}\,\left(\frac{2\,\wp((h\,x + w\,t)\,g_2^{1/4}; 1, 0) - 1}{2\,\wp((h\,x + w\,t)\,g_2^{1/4}; 1, 0) + 1}\right)\right]^{\frac{1}{n-1}} \tag{60}$$

for $A > 0,\ B < 0,\ g_2 > 0$.

- $m = 3\,n - 2$

  In this case, we have also a quartic polynomial

$$P(\varphi) = \frac{2\,A\,(n-1)^2}{n\,(n+1)}\,\varphi + \frac{B\,(n-1)^2}{n\,(2\,n-1)}\,\varphi^4\,. \tag{61}$$

It has two real roots: $\varphi_0 = 0$ and $\varphi_0 = \left(-\frac{2\,A\,(2\,n-1)}{B\,(n+1)}\right)^{1/3}$. For the equation (23), the invariants are

$$g_2 = 0, \qquad g_3 = \frac{-A^2\,B\,(n-1)^6}{4\,n^3\,(n+1)^2\,(2\,n-1)} \tag{62}$$

and $\Delta \neq 0$. Now, the solution of (23) reads from (35) for φ_0,

$$\varphi = \varphi_0 \left[\frac{2\,n\,(n+1)\,\varphi_0\,\wp(\theta;0,g_3) - A\,(n-1)^2}{2\,n\,(n+1)\,\varphi_0\,\wp(\theta;0,g_3) + 2\,A\,(n-1)^2} \right]. \tag{63}$$

Then, the solution of Eq. (1) is from (39) as

$$u(x,t) = \left[\varphi_0 \left(\frac{2\,n\,(n+1)\,\varphi_0\,\wp(h\,x + w\,t;0,g_3) - A\,(n-1)^2}{2\,n\,(n+1)\,\varphi_0\,\wp(h\,x + w\,t;0,g_3) + 2\,A\,(n-1)^2} \right) \right]^{\frac{1}{n-1}} \tag{64}$$

with the conditions: $B < 0$, $g_3 > 0$. Taking into account the relation (38), this solution also can be expressed in terms of the equianharmonic case of the Weierstrass function:

$$u(x,t) = \left[\left(-\frac{2\,A\,(2\,n-1)}{B\,(n+1)} \right)^{1/3} \left(\frac{2^{2/3}\,\wp((h\,x + w\,t)\,g_3^{1/6};0,1) - 1}{2^{2/3}\,\wp((h\,x + w\,t)\,g_3^{1/6};0,1) + 2} \right) \right]^{\frac{1}{n-1}}. \tag{65}$$

We have also plotted these solutions for some special values in Figs. (1)-(5). We can appreciate that for the considered cases, except the parabolic case (42), they consist in periodic waves, some are singular while others are regular. Their amplitude is governed by the non-vanishing constants A, B and their formulas are given in terms of the special forms (36)-(38) of the $\wp$ function.

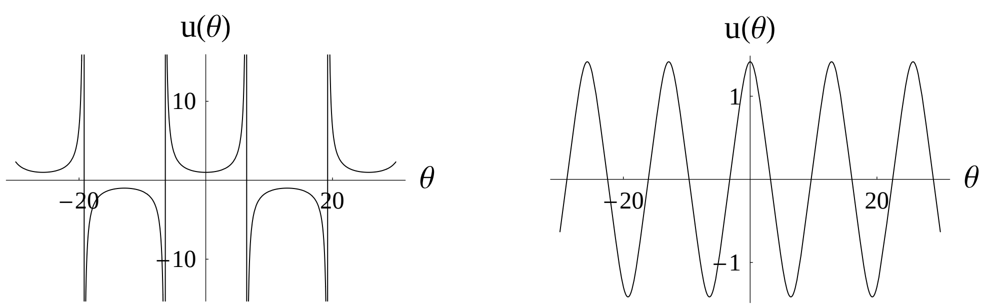


Figure 1: The left figure corresponds to the solution (54) for $h = -2$, $w = 1$, $a = -1$, $n = 3$, $m = 5$ and the right one corresponds to the solution (53) for $h = 1$, $w = 1$, $a = -1$, $n = 3$, $m = 5$.

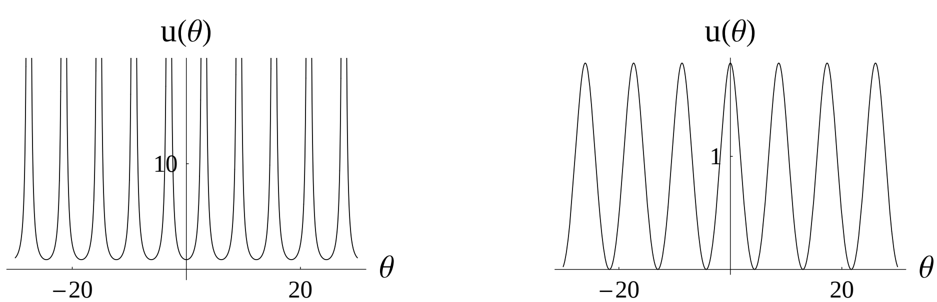


Figure 2: The left figure corresponds to the solution (54) for $h = -2$, $w = 1$, $a = -1$, $n = 2$, $m = 3$ and the right one corresponds to the solution (53) for $h = 1$, $w = 1$, $a = -1$, $n = 2$, $m = 3$.

5 Lagrangian and Hamiltonian

Since Eq. (11) is a motion-type, we can write the corresponding Lagrangian

$$L_W = \frac{1}{2}\,W_\theta^2 + \frac{A\,n}{n+1}\,W^{\frac{n+1}{n}} + \frac{B\,n}{m+n}\,W^{\frac{m+n}{n}} - D\,W \tag{66}$$

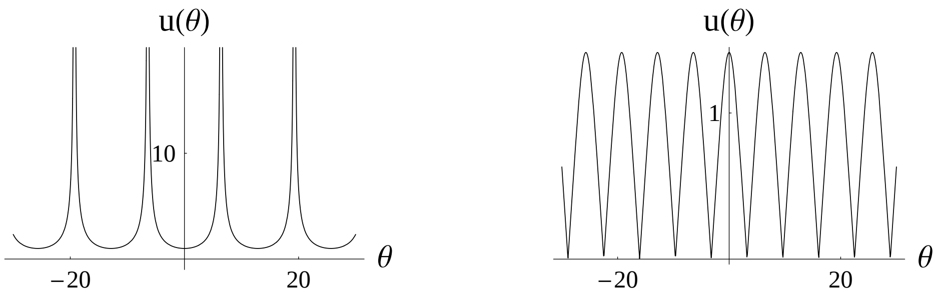


Figure 3: The left figure corresponds to the solution (59) for $h = -2$, $w = 1$, $a = -1$, $n = 3$, $m = 5$ and the right one corresponds to the solution (60) for $h = 1$, $w = 1$, $a = -1$, $n = 3$, $m = 5$.

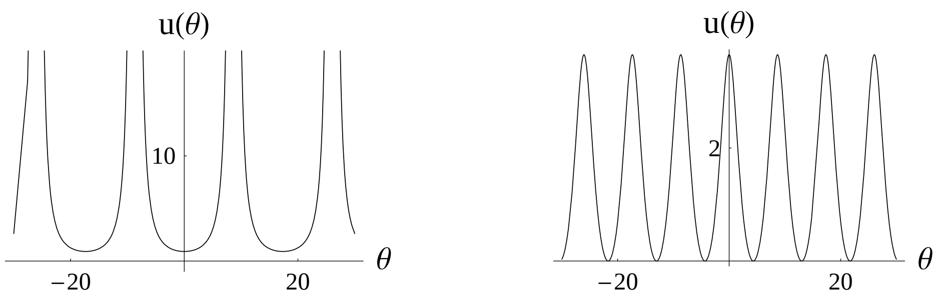


Figure 4: The left figure corresponds to the solution (59) for $h = -2$, $w = 1$, $a = -1$, $n = 2$, $m = 3$ and the right one corresponds to the solution (60) for $h = 1$, $w = 1$, $a = -1$, $n = 2$, $m = 3$.

and, the Hamiltonian $H_W = W_\theta P_W - L_W$ reads

$$H_W(W, P_W, \theta) = \frac{1}{2}\left[P_W^2 - \left(\frac{2\,A\,n}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,B\,n}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W\right)\right] \tag{67}$$

where the canonical momentum is

$$P_W = \frac{\partial L_W}{\partial W_\theta} = W_\theta. \tag{68}$$

The independent variable θ does not appear explicitly in (67), then H_W is a constant of motion, $H_W = E$, with

$$E = \frac{1}{2}\left[\left(\frac{dW}{d\theta}\right)^2 - \left(\frac{2\,A\,n}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,B\,n}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W\right)\right]. \tag{69}$$

Note that this equation also leads to the first order ODE (15) with the identification $C = 2\,E$. Now, the energy E can be expressed as a product of two independent constant of motions

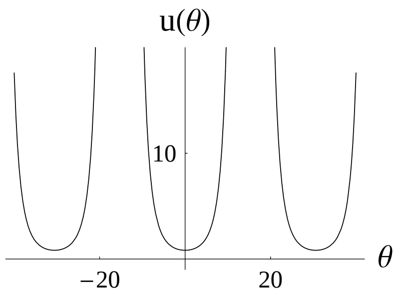
$$E = \frac{1}{2}\,I_+\,I_- \tag{70}$$

where

$$I_\pm(z) = \left(W_\theta \mp \sqrt{\frac{2\,A\,n}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,B\,n}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W}\right)\,e^{\pm S(\theta)} \tag{71}$$

and the phase $S(\theta)$ is chosen in such a way that $I_\pm(\theta)$ be constants of motion ($dI_\pm(\theta)/d\theta = 0$)

$$S(\theta) = \int \frac{A\,W^{\frac{1}{n}} + B\,W^{\frac{m}{n}} - D}{\sqrt{\frac{2\,A\,n}{n+1}\,W^{\frac{n+1}{n}} + \frac{2\,B\,n}{m+n}\,W^{\frac{m+n}{n}} - 2\,D\,W}}\,d\theta. \tag{72}$$



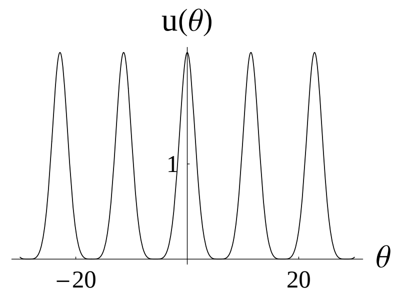


Figure 5: The left figure corresponds to the solution (48) for $h = -2$, $w = 1$, $a = -1$, $n = 2$, $m = 5/2$ and the right one corresponds to the solution (65) for $h = 1$, $w = 1$, $a = -1$, $n = 3/2$, $m = 5/2$.

6 Conclusions

In this paper, we have applied the factorization technique to the $B(m, n)$ equations in order to get travelling wave solutions. We have considered some representative cases of the $B(m, n)$ equation for $m \neq n$. By using this method, we obtained the travelling wave solutions in a very compact form, where the constants appear as modulating the amplitude, in terms of some special forms of the Weierstrass elliptic function: lemniscatic, pseudo-lemniscatic and equiaharmonic. Furthermore, these solutions are not only valid for integer m and n but also non integer m and n. The case $m = n$ for the $B(m, n)$ equations has been examined by means of the factorization technique in a previous paper [14] where the compactons and kink-like solutions recovering all the solutions previously reported have been constructed. Here, for $m \neq n$, solutions with compact support can also be obtained following a similar procedure. We note that, this method is systematic and gives rise to a variety of solutions for nonlinear equations. We have also built the Lagrangian and Hamiltonian for the second order nonlinear ODE corresponding to the travelling wave reduction of the $B(m, n)$ equation. Since the Hamiltonian is a constant of motion, we have expressed the energy as a product of two independent constant of motions. Then, we have seen that these factors are related with first order ODE's that allow us to get the solutions of the nonlinear second order ODE. Remark that the Lagrangian underlying the nonlinear system also permits to get solutions of the system. There are some interesting papers in the literature, where starting with the Lagrangian show how to obtain compactons or kink-like travelling wave solutions of some nonlinear equations [28, 29, 30, 31, 32].

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